Robust trapping at the quartic critical endpoint

Generic forces, centered curvature errors and cumulative drift

Unified Dynamics research workstream | 16 October 2026 UTC
Version 0.1 | Derived conditional theorem

Abstract

The critical endpoint has quartic restoring energy rather than a positive normal Hessian in all directions. We derive an explicit model-error budget from its certified nonlinear tube. Generic gradient errors consume a cubic-in-radius budget, whereas centered errors with zero gradient on the nominal orbit consume a quadratic curvature budget. Static errors yield a nearby compact invariant minimizing set with an explicit cubic-root localization; centered errors improve this to a square-root bound. Cumulative gradient variation gives a time-dependent guarantee. Typed coefficient and loop-readout requirements are supplied. A stiffness increase can satisfy the trapping budget while destabilizing the exact nominal orbit, so trapping and persistence of nominal stability are kept distinct.

For finite $r \geq 1$, $a, N > 0$ and $\epsilon = \kappa/(rN) - k_* = 0$, let R_{p} be the explicit certified radius of [1]. Choose $0 < \eta \leq R_{\mathrm{p}}$ and put

$$c = \frac{N}{1600}, \quad J = \max(a, 25N), \quad B = c\eta^4, \quad \delta = \eta^2\sqrt{c/J}, \qquad \nu_* = \frac{c\eta^3}{8}.$$

Theorem 1 (Endpoint error budget)**.** *Let $V = V_g + V_c$ be a smooth node-gauge/species-phase invariant Hamiltonian error on the fixed-norm manifold. On the closed distance-η tube about the frozen orbit $\mathcal{O}_*$, assume*

$$\|\operatorname{grad} V_g\| \leq \nu, \qquad \operatorname{grad} V_c|_{\mathcal{O}_*} = 0, \qquad \|\operatorname{Hess} V_c\| \leq \beta, \qquad \boxed{\nu + \beta\eta \leq \nu_*.}$$

Every initial state at orbit distance at most δ stays at distance less than η for all real time under $H + V$. On zero Gauss there is a nonempty compact invariant minimizing set, stable as a set, with

$$d_g(x, \mathcal{O}_*) \leq r_+(\nu, \beta) \leq \eta/2, \quad cr_+^3 - \frac{\beta}{2}r_+ - \nu = 0,$$

where r_+ is the largest nonnegative root. For purely centered errors,

$$\boxed{\beta \leq \frac{c\eta^2}{8}, \qquad r_+ = \sqrt{\frac{\beta}{2c}} \leq \frac{\eta}{4}.}$$

Status. Development, no registry entry. Frozen action and ledger unchanged. Constructor verification only; independent review pending; physical promotion 0. The theorem does not assert a unique shifted orbit, attraction or retained stability of the nominal orbit.

1 Model, metric and the inherited quartic tube

The frozen tetrahedral Hasse carrier has 15 nodes, 28 incidence links and 18 diamonds. For link $e = (i \leftarrow j)$, set $D_U[i,j] = b_e U_e$, $b_e = B_{ij} = \pm 1$, $h_U = i(D_U^\dagger - D_U)$ and

$$W_d = (U_{ih}U_{hk})^\dagger U_{ij}U_{jk}, \qquad d = (i,j,h,k).$$

With $q = \psi/\sqrt{N}$, $w = \bar{\chi}/\sqrt{N}$, $\|q\| = \|w\| = 1$,

$$H = \frac{a}{2}\sum_e \operatorname{tr} E_e^2 + \kappa \sum_d \left(1 - \frac{1}{r}\operatorname{Re}\operatorname{tr} W_d\right) + N(q^\dagger h_U q + w^\dagger h_U w) + 2\mu N.$$

The two species have opposite symplectic forms. The metric is the product of unnormalized Frobenius bi-invariant link metrics, Euclidean Frobenius electric metrics and real round metrics on the two unit matter spheres:

$$\|(iA_eU_e, F_e, p, z)\|_g^2 = \sum_e \|A_e\|_F^2 + \sum_e \|F_e\|_F^2 + \|p\|^2 + \|z\|^2.$$

The critical representative is $U_e^* = \operatorname{diag}(-ib_e, I_{r-1})$, $E^* = 0$, $q_i^* = w_i^* = v_i e_1$. Here $v > 0$ is the unit Perron eigenvector of unsigned adjacency, $\rho = \sqrt{8 + 2\sqrt{10}}$, $v_i > 1/5$, and $k_* \simeq 0.03760248457$. The energy reference is $H_* = 2N(\mu - \rho) + 36Nk_*$. The compact gauge/species-phase orbit $\mathcal{O}_*$ includes the relative phase motion.

Write $d(x) = d_g(x, \mathcal{O}_*)$. The PET theorem [1] supplies

$$cd(x)^4 \le H(x) - H_* \le \frac{J}{2}d(x)^2 \quad (d(x) \le R_{\mathrm{p}}), \qquad \|\operatorname{grad} H(x)\| \le Jd(x). \tag{1}$$

The lower bound uses intrinsic parity and a sixth-order remainder; the upper and gradient estimates follow from a global Hessian bound. A fully explicit radius is obtained by

$$\alpha = \min(a/2, N/100), \quad D_0 = 279N/(4\alpha), \quad C_3 = 279N,\ C_4 = 1600N,\ C_5 = 10000N,$$

$$S_6 = \frac{9C_3^2D_0^2}{2\alpha} + \frac{C_3D_0^3}{6} + \frac{C_4^2}{18\alpha} + 5C_4D_0^2 + \frac{2C_5D_0}{15} + \frac{8C_5^2}{9\alpha} + 264N,$$

$$R_{\mathrm{p}} = \min\left\{\frac{\alpha}{500000N}, \sqrt{\frac{N}{400S_6}}\right\} \le 2 \times 10^{-8}.$$

The unchanged helper returns an inward-rounded rational radius. These are mathematical metric/energy units. No laboratory noise or physical time scale is assigned.

Spatial errors are at least C^2 near the closed tube and preserve the stated gauge and independent species-phase symmetries. They therefore preserve the fixed species norms and matrix moment map. The zero-Gauss subset remains invariant; its final quotient can be singular. No free quotient chart is assumed.

2 Static energy transfer and minimizing-set localization

An invariant error is constant on $\mathcal{O}_*$. Subtract its orbit value and write $W = V - V(\mathcal{O}_*)$. A shortest geodesic from the orbit to a point at distance $d \le \eta$ stays in the tube. The hypotheses give

$$|W(x)| \le \nu d(x) + \frac{\beta}{2} d(x)^2, \qquad \|\operatorname{grad} V(x)\| \le \nu + \beta d(x) \le \nu_*. \tag{2}$$

The centered estimate integrates its Hessian from a zero starting gradient. Gauge invariance transports that condition over the entire orbit.

Trapping part of Theorem 1. Let $s = d(x_0) < \eta$ and $E_0 = H(x_0) - H_*$. Conservation of $H + V$ gives, until first exit,

$$H(x(t)) - H_* \le E_0 + \nu(s + \eta) + \frac{\beta}{2}(s^2 + \eta^2).$$

A strict upper bound below B therefore suffices for no exit. If $s \le \delta$, then $E_0 \le J\delta^2/2 = B/2$, and $\delta/\eta = \eta\sqrt{c/J} \le \eta/200 < 1$. The total potential variation is at most

$$2\nu\eta + \beta\eta^2 \le 2\eta(\nu + \beta\eta) \le B/4.$$

The nominal energy stays at most $3B/4$, while reaching $d = \eta$ would require at least B by (1). This is impossible in either time direction. Compact trapping and smoothness give complete flow for these initial states. □

Minimizing-set part. Intersect the closed tube with the zero-Gauss fixed-norm surface. This is compact, contains $\mathcal{O}_*$, and need not be a smooth manifold. Minimize $F = H - H_* + W$ there. On the boundary,

$$F \ge B - \nu\eta - \beta\eta^2/2 \ge 7B/8 > 0,$$

whereas $F = 0$ on the nominal orbit. The full nonempty compact minimizing set $\mathcal{K}$ lies in the interior. At any minimizer,

$$cd^4 - \nu d - \beta d^2/2 \le F \le 0.$$

For $d > 0$, this gives $cd^3 - \beta d/2 - \nu \le 0$ and hence $d \le r_+$. The cubic is positive beyond its largest nonnegative root. Moreover

$$c(\eta/2)^3 - \beta\eta/4 - \nu \ge 3\beta\eta/4 \ge 0,$$

so $r_+ \le \eta/2$. If $\nu = 0$, its largest root is $\sqrt{\beta/(2c)} \le \eta/4$; if $\beta = 0$, it is $(\nu/c)^{1/3}$.

Energy conservation and the boundary gap keep every trajectory from $\mathcal{K}$ inside it. For any sufficiently small neighborhood of the full set $\mathcal{K}$, its compact complement in the tube has strictly higher F. This gap proves stability of $\mathcal{K}$ as a set. A unique equilibrium, retained degeneracy or attraction does not follow. □

3 Time-dependent drift and the role of cumulative control

For time-dependent invariant V_t, set

$$W_t(x) = V_t(x) - V_t(\mathcal{O}_*), \qquad F_t = H - H_* + W_t.$$

This removes arbitrary time-dependent constants. Suppose the amplitude bounds of Theorem 1 hold uniformly, so $\|\operatorname{grad} V_t\|_{\text{tube}} \le \nu_*$. Let $\rho(t)$ bound $\|\operatorname{grad} \partial_t V_t\|$ on that tube. Require

$$\boxed{\int_0^T \rho(t)\,dt \le \nu_*.}$$

Along the perturbed Hamiltonian flow, $\dot{F}_t = \partial_t W_t$ exactly. Since $|\partial_t W_t| \le \rho(t)d$, for $s \le \delta$ before first exit,

$$H(x(t)) - H_* \le \frac{J}{2}s^2 + \nu_* s + \nu_* \eta + \eta \int_0^T \rho(t)\,dt \le \frac{7B}{8} < B.$$

Thus trapping holds through T. A bound on the total integral over $[0, \infty)$ gives all-forward-time trapping. No tracking of instantaneous minimizers is implied.

For a differentiable split $V_t = V_{g,t} + V_{c,t}$ with centered $V_{c,t}$ at every time, it suffices that

$$\rho(t) \le \nu_1(t) + \eta\beta_1(t),$$

where ν_1 bounds $\|\operatorname{grad} \partial_t V_{g,t}\|$ and β_1 bounds $\|\operatorname{Hess} \partial_t V_{c,t}\|$. These are bounds on coefficient derivatives or potential derivatives, not derivatives of a previously reported bound envelope.

Why amplitude alone is insufficient. Even a system satisfying a quartic lower bound on a tube can escape under arbitrarily small bounded periodic driving. On $q^2 + p^2 \le 1$,

$$H_0 = (q^2 + p^2)/2 \ge (q^2 + p^2)^2/2.$$

For $V_t = -fq\sin t$, $f > 0$ can be arbitrarily small, with $\|\operatorname{grad} V_t\| \le f$. The exact zero-initial-state solution is

$$q(t) = \tfrac{f}{2}(\sin t - t\cos t), \qquad p(t) = \tfrac{ft}{2}\sin t.$$

At $t = 2\pi n$ it has $q = -\pi f n$, $p = 0$, and leaves every fixed radius. The constructor replay verifies the ODE, initial values and sample sequence. This is a counterexample within the abstract quartic-coercivity class, not a claim that the same drive destabilizes every UD orbit. The cumulative condition excludes this generic inference from instantaneous smallness.

No non-Hamiltonian force is included in the present budget. If such a tangent field is later admitted, its relative-energy work must be added explicitly. A physical experiment would also need a calibrated mapping from the model evolution parameter to time.

4 Typed coefficient budgets at the critical orbit

Admit kinetic errors Δa_e, diamond errors $\Delta\kappa_d$, real relative hopping errors ξ_e and real scalar site detunings v_i:

$$V = \frac{1}{2}\sum_e \Delta a_e \operatorname{tr} E_e^2 + \sum_d \Delta\kappa_d \left(1 - \frac{1}{r}\operatorname{Re}\operatorname{tr} W_d\right) + N(q^\dagger \delta h_U q + w^\dagger \delta h_U w) + N\sum_i v_i(\|q_i\|^2 + \|w_i\|^2),$$

where $\delta D_U[i,j] = b_e \xi_e U_e$ and $\delta h_U = i(\delta D_U^\dagger - \delta D_U)$. Choose a real common component ξ_0 and define

$$A = \max|\Delta a_e|, \quad K = \max_d \frac{|\Delta\kappa_d|}{rN}, \quad H_c = |\xi_0|, \quad G_r = \max_e |\xi_e - \xi_0|, \quad D_s = \frac{\max_i v_i - \min_i v_i}{2}.$$

For real hopping errors, the midpoint of their range is one possible common component. For time variation choose a differentiable component or bound its piecewise derivatives almost everywhere.

Lemma 2 (Centered coefficient split)**.** *The kinetic, diamond and common real hopping errors form a centered V_c with*

$$\|\operatorname{Hess} V_c\| \le \beta_{\text{coeff}} = A + 12NK + 24NH_c.$$

The residual hopping and site errors form V_g with

$$\|\operatorname{grad} V_g\| \le \nu_{\text{coeff}} = 14NG_r + 3ND_s.$$

Consequently a sufficient mixed requirement is

$$\boxed{14NG_r + 3ND_s + \eta(A + 12NK + 24NH_c) \le \frac{N\eta^3}{12800}.}$$

Proof. The kinetic gradient vanishes at $E = 0$ and its global Hessian norm is at most A. Every background Wilson matrix is Hermitian, $\operatorname{diag}(-1, I)$, so each diamond potential has zero first derivative there, individually. Two-generator trace estimates, four links per diamond and at most three diamonds per link give Hessian norm at most $12NK$ independently of finite rank.

A common real hopping error multiplies the nominal hopping functional. Its matter derivative vanishes on the fixed norms because v is a hopping eigenvector; its link derivative vanishes at the real-ground critical phase. Its global second geodesic derivative is bounded by $24NH_c$, from the degree-four block Schur and sphere-speed estimate in [1]. These centered conditions transport over $\mathcal{O}_*$ by invariance.

For residual hopping, degree four bounds its hopping operator and link derivative by $4G_r$ and $4G_r b$. Along a unit tangent with link/matter speeds b, c_1, c_2, the energy derivative is at most $8NG_r(b + c_1 + c_2) \le 8\sqrt{3}NG_r < 14NG_r$. Subtracting the uniform midpoint of the site detunings leaves operator norm at most D_s and derivative at most $2ND_s(c_1 + c_2) \le 2\sqrt{2}ND_s < 3ND_s$. Uniform site or mass shifts are constant on the norm surface and spend no budget. □

Complex hopping errors can be covered generically by choosing $\xi_0 = 0$ and $G_r = \max|\xi_e|$; only the centered common component must be real. For time variation use the same bounds on coefficient derivatives to construct $\nu_1 + \eta\beta_1$, then integrate it separately from the amplitude bound.

5 Displacement exponents and nominal-orbit instability

The minimizing-set localization is Hölder rather than Lipschitz at the quartic endpoint. Its exponents cannot generally be improved using only the stated coercivity and derivative hypotheses.

For $H_0 = p^2/2 + cq^4$ and a generic tilt $V = -\nu q$, the unique minimum is

$$p = 0, \qquad q = (\nu/(4c))^{1/3}.$$

The error gradient has norm ν and the displacement scales exactly as $\nu^{1/3}$. For a centered error $V = -\beta q^2/2$, the two minima are

$$p = 0, \qquad q = \pm\sqrt{\beta/(4c)}.$$

Here $\operatorname{grad} V(0) = 0$, $\|\operatorname{Hess} V\| = \beta$, and displacement scales as $\sqrt{\beta}$. The origin remains stationary and becomes a saddle for every $\beta > 0$. These are exact normal-form counterexamples to stronger generic persistence claims, not a proof that every UD coefficient perturbation attains these constants.

An admitted UD stiffness shift. Let the normalized magnetic coefficient increase uniformly by $t > 0$, so $\Delta\kappa_d = rNt$ and $K = t$. Its gradient at the critical orbit is zero and its Hessian bound is $\beta = 12Nt$. It passes the present sufficient contract whenever

$$0 < t \le \frac{\eta^2}{153600}.$$

Yet it places the original orbit above the frozen sharp stability boundary [3]. Explicitly, in a pure active critical phase direction Zb, its additional quadratic energy coefficient is

$$-\frac{Nt}{2}\|CZb\|^2 < 0 \quad (b \neq 0),$$

where C is the diamond curl and CZ has rank three in the frozen kernel identity. The original relative equilibrium remains stationary, but the inherited dynamical theorem makes it unstable above the boundary. The minimizing-set result still places the full new minimizing set within

$$d \le \sqrt{\frac{12Nt}{2c}} = \sqrt{9600t} \le \eta/4.$$

Thus finite-radius trapping does not imply stability of the exact nominal orbit under parameter error. Instability can produce departure from a smaller neighborhood while the trajectory remains inside the larger certified tube. No contradiction or attraction to a new minimum is implied.

This is why the theorem concludes existence and stability of the complete minimizing invariant set, rather than uniqueness of a shifted equilibrium or survival of the endpoint degeneracy. The frozen nonlinear phase-branch diagram is separate conditional support, not a new independently reviewed bifurcation result in this release.

6 Readout transfer and mathematical experiment requirements

For a diamond based at node k, retain the gauge-invariant normalized readout

$$z_d(x) = -\frac{q_k^\dagger W_d q_k}{\|q_k\|^2}.$$

At the critical orbit $z_d^* = 1$ for all 18 diamonds. For $d(x) \le \eta \le R_{\mathrm{p}}$, each denominator stays nonzero and

$$\boxed{|z_d(x) - 1| \le \sqrt{404}\, d(x) < 21\eta.}$$

To verify the bound, align with a closest orbit representative. The nominal node norm exceeds $1/5$ and its matter chord error is at most global round distance. Normalization changes the local unit vector by at most ten times that matter error, contributing a coefficient twenty to the expectation. Four-link telescoping contributes coefficient two times the joint link distance. Cauchy–Schwarz gives $\sqrt{20^2 + 2^2} = \sqrt{404} < 21$ [2]. Since $R_{\mathrm{p}} < 1/10$, local matter cannot vanish.

An instrument with complex per-channel error at most σ has total error at most $21\eta + \sigma$ relative to the nominal readout. For a declared target $\tau > 0$, it suffices to choose

$$\eta \le \min(R_{\mathrm{p}}, \tau/42), \qquad \sigma \le \tau/2.$$

The nominal endpoint readouts coalesce; no 26-branch classifier is supplied here. This readout bound follows from a state bound. It does not conversely certify the full initial orbit distance, which includes electric and matter variables not recovered by these loops alone.

At $a = N = 1$ and $\eta = 1/829538022$, the canonical sufficient values are

Quantity	Approximate value
Target tube η	1.20549×10^{-9}
Initial radius δ	7.26603×10^{-21}
Boundary energy B	1.31988×10^{-39}
Generic gradient cap ν_*	1.36861×10^{-31}
Pure centered Hessian cap ν_*/η	1.13532×10^{-22}
Pure normalized magnetic error cap K	9.46098×10^{-24}
Complex loop state-error bound 21η	2.53153×10^{-8}

The coefficient caps are alternatives or shared reserves, not independent simultaneous allowances. The exact calculator accepts a mixed allocation and a separate cumulative variation input. Zero defaults are illustrative assumptions, not measurements. Out of budget means the sufficient test is inconclusive, not that instability has been proved.

A proposed experiment must specify a realization of the carrier/action; calibrated state, energy, time and detector maps; certified preparation distance or relative energy; coefficient amplitude and cumulative rate bounds; and instrument uncertainty. The attached machine-readable contract records these mathematical requirements. None is a physical identification or a substitute for independent review.

7 Verification, provenance and remaining scope

Exact constructor replay. With Python 3 and SymPy, run

```
python3 verify_endpoint_robustness.py
```

The certificate records 24 exact algebraic checks: energy allocation, the mixed cubic localization identity, coefficient/readout constants, the two quartic normal forms, the stiffness shift and the resonant-drive obstruction. Five exact rational budget examples include generic-only, centered-only, mixed and unequal-coefficient cases. A rational bisection encloses the largest nonnegative localization root. The frozen PET certificate is hash-bound as inherited support and is not counted as new interval evidence or independent review.

Synthetic diagnostic. With NumPy, SciPy and python-flint 0.9.0, run

```
python3 check_endpoint_gradients.py
```

Four deterministic full-matrix fixtures at $r = 1, 2, 3, 5$ check centered gradients at the critical representative, their linear growth on a short geodesic, the generic/mixed coefficient bounds, finite-difference directional derivatives, free uniform detuning and loop readouts. Maximum derivative discrepancy is below 8×10^{-13} in these fixtures. Unscaled coefficient shapes are reported with the factor needed to fit the canonical tube budget. Gauss is not imposed because the analytic inequalities hold on the larger fixed-norm surface. These are numerical diagnostics, not empirical samples or a proof over a domain.

Calculator. Uses the Python standard library and the unchanged PET radius helper:

```
python3 endpoint_robustness_budget.py -a 1 -N 1
```

Optional exact-rational inputs set the target radius, five typed coefficient bounds and cumulative variation. The variation input must bound the integrated spatial gradient of the time derivative, using the stated coefficient-rate formula. The calculator command help lists the options. Decimal fields are diagnostics; radius, energy, initial-radius square and root enclosures are exact rational outputs.

Frozen references (same constructor).

1. PET v0.1, *Parity-sharpened trapping at the critical endpoint*, 16 October 2026. UD portal G17, PR 165, capture suffix `37748fe1`. Supplies the quartic tube and its frozen dependencies.
2. RHT v0.1, *Robust nonlinear trapping under Hamiltonian model errors*, 16 October 2026. UD portal G15. Positive-offset transfer method and matrix/readout estimates; the present endpoint proof is derived explicitly above.
3. HRC v0.1, *Higher-rank curved stability theorem*, and CBF v0.1, *Curved-branch bifurcation theorem*, 15 October 2026. UD portal G8 and G10. Supplies the sharp nominal parameter boundary and phase-branch context.

The action and coefficient ledger are unchanged. Development, no registry entry; independent review pending; physical promotion 0. Conditional dependency depth is retained. No unique perturbed orbit, attraction, sharp basin, infinite-rank flow, physical coupling unification or proton/monopole/neutrino confirmation is established. No experimental samples or unavailable image are proof inputs. The next bridge task needs a specified realization and calibrated observation map before a data reanalysis can test this model.