Gauge-Covariant Incidence Transport on a Filled Tetrahedron

A flatness theorem and an exact phase-controlled spectrum

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AI-assisted development manuscript

15 October 2026 UTC / version 0.1

EXACT CONDITIONAL mathematics. CANDIDATE gauge extension.
Constructor verification passed; independent review pending. Physical promotion: **0**.

Abstract

We construct a finite unitary transport extension of the declared tetrahedral incidence flow. A rank-r complex fiber is assigned to each simplex and an independent unitary map to each adjacent incidence. The resulting generator is skew-Hermitian, has gauge-invariant support currents, and is covariant under local frame changes and signed tetrahedral relabelings. Its degree-two square is the signed difference of transport along the two sides of an incidence diamond. On the filled tetrahedron, preserving the chain identity forces all such curvature to vanish and the connection to be pure gauge. Curved transport remains norm conserving but changes the original Hodge-square identity. A single phase on one vertex–edge incidence gives an exactly solvable spectral splitting, providing a finite analogue test contract. The construction supplies a precise candidate interface, not a derivation of the Standard Model, a gauge-field action, a continuum limit, running couplings or physical unification. The underlying connection/curvature principles are established mathematics; no general novelty claim is made.

1 Source commitments and added objects

The source carrier is the downward-closed filled tetrahedron Δ^3, with increasingly oriented simplices and graded dimensions $(4, 6, 4, 1)$. Let B be the total boundary, acting downward on the direct sum of all 15 coordinates. The native real flow is

$$B^2 = 0, \qquad A_0 = B^T - B, \qquad \dot{c} = A_0 c.$$

Its complexification and unit simplex metric are mathematical choices, not a physical quantization. Papers I–III, review drafts 1.1.1, retain their source identities [1, 2, 3]. In particular, the finite S_4 symmetry is not itself a continuous physical gauge group.

The Hasse graph has one node per nonempty simplex and one link for each adjacent containment. It has 15 nodes and 28 links. We add the following:

Object	Type and status
$E_\sigma = \mathbb{C}^r$	Added internal fiber at simplex σ; r is supplied.
$c_\sigma \in E_\sigma$	Complex amplitude vector, with support $q_\sigma = \|c_\sigma\|^2$.
$U_{\sigma\tau}$	Unitary map $E_\tau \to E_\sigma$ for $\sigma \subset \tau$ of codimension one. Reverse transport is its adjoint.
s	Internal evolution parameter; no laboratory clock is assigned.
$\Omega_\sigma(s)$	Optional local skew-Hermitian temporal connection.

The fibers are indexed by simplices, not merely by the four geometric vertices. Thus the local frame group is $\prod_\sigma \mathrm{U}(r)$. This is an explicit extension with additional state data. It is not a discovery that the original scalar coordinate already carries color, weak isospin or hypercharge.

Define a downward block operator by

$$(D_U)_{\sigma\tau} = B_{\sigma\tau} U_{\sigma\tau}, \qquad A_U = D_U^\dagger - D_U, \qquad L_U = D_U^\dagger D_U + D_U D_U^\dagger. \tag{1}$$

Unlisted blocks vanish. For identity transport, $D_U = B \otimes I_r$. The original topological boundary B remains defined even when the new transport operator D_U is not nilpotent.

2 The transport and flatness theorem

A codimension-two containment $\sigma \subset \eta$ has exactly two intermediate simplices τ, τ'. Write $\epsilon = B_{\sigma\tau} B_{\tau\eta}$; the other path has sign $-\epsilon$. Its transport defect and based diamond holonomy are

$$F_{\sigma\eta} = U_{\sigma\tau} U_{\tau\eta} - U_{\sigma\tau'} U_{\tau'\eta}, \qquad W = (U_{\sigma\tau'} U_{\tau'\eta})^\dagger U_{\sigma\tau} U_{\tau\eta}. \tag{2}$$

Both are independent of the matter state. F maps E_η to E_σ; W is an endomorphism of E_η.

Theorem 1 (Finite gauge-transport admissibility)**.** *On the filled tetrahedron, for every integer $r \geq 1$ and every assignment of unitary incidence transports, the following statements hold.*

1. *$A_U^\dagger = -A_U$. For continuous prescribed links, the equation $\dot{c} = (A_U + \Omega)c$ preserves total support when $\Omega = \bigoplus_\sigma \Omega_\sigma$ is skew-Hermitian.*
2. *Under $c'_\sigma = g_\sigma c_\sigma$ and $U'_{\sigma\tau} = g_\sigma U_{\sigma\tau} g_\tau^\dagger$, $D_{U'} = G D_U G^\dagger$ and $A_{U'} = G A_U G^\dagger$. For differentiable time-dependent frames, full evolution covariance requires $\Omega' = G\Omega G^\dagger + \dot{G} G^\dagger$.*
3. *Every codimension-two block is $(D_U^2)_{\sigma\eta} = \epsilon F_{\sigma\eta}$. Consequently $D_U^2 = 0$ if and only if every diamond holonomy is identity.*
4. *If $D_U^2 = 0$, there are unitary P_σ such that $U_{\sigma\tau} = P_\sigma P_\tau^\dagger$. Therefore A_U is unitarily conjugate to $A_0 \otimes I_r$ and*
$$\det(\lambda I - A_U) = \lambda^r (\lambda^2 + 4)^{7r}. \tag{3}$$
5. *For arbitrary links,*
$$-A_U^2 = L_U - D_U^2 - (D_U^\dagger)^2. \tag{4}$$
Thus the unchanged degree-diagonal Hodge-square identity $-A_U^2 = L_U$ holds if and only if the connection is flat in the sense of item 3.

Proof. Adjoint transposition of (1) proves skew-Hermiticity. With $Q = \sum_\sigma c_\sigma^\dagger c_\sigma$, differentiation gives $\dot{Q} = c^\dagger[(A_U + \Omega)^\dagger + (A_U + \Omega)]c = 0$. Each nonzero block transforms by the stated left/right frame factors; summing proves covariance. Differentiating $c' = Gc$ supplies the inhomogeneous $\dot{G} G^\dagger$ term. Omitting it is valid only for time-independent frames, or a separately fixed temporal gauge.

For item 3, a two-step downward path from η to σ removes two distinct vertices. Exactly two orders are possible, and the ordinary incidence signs are opposite. Their two transport products are therefore precisely $\epsilon F_{\sigma\eta}$. All other blocks of D_U^2 vanish by degree. Since the second path product is unitary, $F = 0$ is equivalent to $W = I$.

For item 4 let t denote the unique top tetrahedron. Choose $P_t = I$ and define $P_\sigma : E_t \to E_\sigma$ by transport along a downward path from t to σ, without its incidence signs. Such paths correspond to orders of deleting the vertices absent from σ. Adjacent transpositions generate all changes of

order; each transposition is exactly a diamond. Flatness makes the path product independent of the order. In particular $P_\sigma = U_{\sigma\tau} P_\tau$ for each incidence. All P_σ are unitary, so

$$D_U = P(B \otimes I_r)P^\dagger, \qquad P = \bigoplus_\sigma P_\sigma.$$

This proves the conjugacy and (3), using the native spectrum. The unique top cell is substantive: flat connections on a non-simply-connected base need not be globally pure gauge.

Finally expand $(D_U^\dagger - D_U)^2$ to obtain (4). The two correction terms lower and raise degree by two respectively. They occupy disjoint graded blocks, so their sum vanishes only when each vanishes. This proves item 5. □

The flatness conclusion concerns incidence transport at each fixed s. It does not exclude mixed temporal curvature

$$\mathcal{E}_{\sigma\tau} = \dot{U}_{\sigma\tau} - \Omega_\sigma U_{\sigma\tau} + U_{\sigma\tau}\Omega_\tau,$$

which transforms as $g_\sigma \mathcal{E}_{\sigma\tau} g_\tau^\dagger$. For example, identity links with unequal local scalar temporal blocks can have $\mathcal{E} \neq 0$ while $D_U^2 = 0$. Instantaneous conjugacy of A_U therefore does not establish equality of the complete time-dependent propagators. The spectral corollary below fixes $\Omega = 0$ and constant links.

2.1 Support currents and full tetrahedral covariance

For each adjacent incidence put

$$J_{\sigma\tau} = 2\,\mathrm{Re}(c_\sigma^\dagger B_{\sigma\tau} U_{\sigma\tau} c_\tau). \tag{5}$$

Its contribution is $-J_{\sigma\tau}$ to $\dot{q}_\sigma$ and $+J_{\sigma\tau}$ to $\dot{q}_\tau$. Local temporal blocks contribute zero to each q_σ. Thus for any coordinate region R,

$$\frac{d}{ds} \sum_{\sigma \in R} q_\sigma = J_{\mathrm{in}}(R) - J_{\mathrm{out}}(R).$$

Equation (5) is gauge invariant because both endpoint frame factors cancel. It is a model support current, not an electrical-current identification.

For $\pi \in S_4$ let S_π be the signed permutation of oriented simplices. Move the independent link data by $U^\pi_{\pi\sigma,\pi\tau} = U_{\sigma\tau}$. Incidence equivariance gives

$$D_{U^\pi} = (S_\pi \otimes I_r) D_U (S_\pi^T \otimes I_r). \tag{6}$$

This is covariance of the whole connection family. A particular nonuniform connection need not commute with S_π. No curved configuration is declared tetrahedrally invariant merely because the law is covariant.

For a diamond a dimensionless gauge-invariant curvature diagnostic is

$$\chi_W = 1 - \frac{1}{r}\,\mathrm{Re}\,\mathrm{tr}\, W = \frac{\|F_{\sigma\eta}\|_F^2}{2r} \in [0, 2]. \tag{7}$$

This follows by expanding the Frobenius norm of the difference of the two unitary path products. On the scalar graph, a spanning tree removes 14 of the 28 link phases, leaving 14 independent loop coordinates. The 18 diamond equations generate all loop constraints here by the path proof above. The replay also checks their linearized scalar rank, 14. These are model coordinate counts, not physical particle or spacetime dimensions.

3 An exact, phase-controlled spectral prediction

Set $r = 1$, $\Omega = 0$, all links to one except $U_{\{0\},\{0,1\}} = u = e^{i\phi}$. Keep every link magnitude and every incidence sign fixed.

Corollary 2 (One-incidence phase spectrum)**.** *For this declared family,*

$$\det(\lambda I - A_U) = \lambda(\lambda^2+4)^5\left[(\lambda^2+4)^2 - 4(1-\cos\phi)\right]. \tag{8}$$

The positive squared frequencies are 4 *five times and* $4\pm 2\sqrt{1-\cos\phi}$ *once each, with multiplicities combined at* $\phi = 0$*. At* $u = i$*,*

$$\det(\lambda I - A_U) = \lambda(\lambda^2+2)(\lambda^2+4)^5(\lambda^2+6). \tag{9}$$

The connection is curved for $u \neq 1$ *even though all support balances remain exact. At* $u = i$*,* $\operatorname{rank} D_U^2 = 1$ *and* $\operatorname{rank}(-A_U^2 - L_U) = 2$*.*

Proof. Partition the 15 coordinates into eight even-degree and seven odd-degree coordinates. In this order write $A_U = \left(\begin{smallmatrix} 0 & C \\ -C^\dagger & 0 \end{smallmatrix}\right)$. For the identity-link matrix C_0, the native identity gives $C_0^\dagger C_0 = 4I_7$. The vertex-0 row has three unit entries. Let e select odd coordinate edge 01 and let $v = C_0^\dagger e_{\text{vertex }0} = e + w$, so $w^\dagger e = 0$ and $\|w\|^2 = 2$. With $\delta = u - 1$,

$$C = C_0 + \delta e_{\text{vertex }0} e^\dagger, \qquad C^\dagger C = 4I_7 + \delta w e^\dagger + \bar{\delta} e w^\dagger.$$

The diagonal $ee^\dagger$ correction cancels since $\delta + \bar{\delta} + |\delta|^2 = 0$. On the plane spanned by $e, w/\sqrt{2}$, the added Hermitian matrix has eigenvalues $\pm\sqrt{2}|\delta| = \pm 2\sqrt{1-\cos\phi}$. It is zero on the other five directions. The nonzero eigenvalues of the bipartite skew matrix are the signed imaginary singular values of C, with one additional zero direction. This proves (8). For $u \neq 1$, D_U^2 has only the row at vertex 0, with nonzero entries from faces 012 and 013, so its rank is one. Its adjoint occupies the opposite graded block, proving the defect rank. □

The corollary is a finite-model testable prediction, not an electroweak or strong-coupling spectrum. Frequencies and their ratios are insensitive to a common positive clock multiplier, but an actual sensor must still be calibrated.

4 What the native symmetries do and do not select

Proposition 3 (Global symmetry versus local gauge data)**.** *The real orthogonal centralizer of the canonical native* A_0 *is* $O(1) \times \mathrm{U}(7)$*. Its subgroup also commuting with the full signed* S_4 *action is* $O(1) \times \mathrm{U}(1)^3$*. Consequently the connected* S_4*-compatible global symmetry has no nontrivial continuous homomorphism from* $SU(2)$ *or* $SU(3)$*.*

Proof. The harmonic line is $\ker A_0$ and must be preserved. On its orthogonal complement $J = A_0/2$ is an orthogonal complex structure. Orthogonal maps commuting with J are exactly U(7). As a complex S_4 representation this space is the sum of one copy each of the inequivalent real-type irreducibles of complex dimensions $1, 3, 3$. Schur's lemma gives one unit phase per summand. A homomorphism from either connected simple group to the abelian torus has zero Lie algebra map, hence trivial image. □

This result does not exclude differently embedded, noncommuting or enlarged gauge branches. In particular U(7) contains non-Abelian subgroups if the simultaneous S_4-commuting restriction

is dropped. It supplies neither a chosen embedding nor local parallel transport. The requirement of commuting with S_4 is an explicit branch assumption, not a universal law.

For a separate comparison, exact scalar state phases $z_\sigma = c_\sigma/|c_\sigma|$ on nonzero support produce receipts $u_{\sigma\tau} = \bar{z}_\sigma z_\tau$. Their product around every closed support-graph path is one by telescoping, with conjugation on reversed edges. Zeros remove links and require separate components. Therefore these relative-state receipts cannot alone instantiate an independent nonflat connection. They remain valid reconstruction data as in Paper II; no source result is withdrawn.

5 Coupling selection remains a separate theorem target

Even after *assuming* the Standard Model Lie algebra $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, gauge invariance does not impose a common kinetic coefficient. Every positive invariant symmetric bilinear form is

$$h = a_3 h_3 \oplus a_2 h_2 \oplus a_1 h_1, \qquad a_i > 0, \tag{10}$$

once generator normalizations are fixed. On each compact simple ideal the adjoint representation and Schur's lemma give one scalar multiple of a fixed invariant form. Cross terms vanish because each simple ideal is perfect and commutes with the other ideals and the center. The one-dimensional center has its independent positive coefficient. This proves the three-parameter freedom without fitting a numerical target.

A chosen compact simple algebra instead has one such form up to scale. Conditional $SU(5)$ illustrates the normalization obligation: if one supplies

$$Y = \mathrm{diag}(-1/3, -1/3, -1/3, 1/2, 1/2), \quad \mathrm{tr}\, Y^2 = 5/6, \quad T_Y = \sqrt{3/5}\, Y,$$

then $\mathrm{tr}\, T_Y^2 = 1/2$ and a common $g_5 T_Y = g_Y Y$ gives $g_1 = \sqrt{5/3}\, g_Y = g_5$ at tree-level matching. This is standard conditional group theory [6], not selection of $SU(5)$ by UD and not a running-coupling calculation. A renormalization scheme, matter spectrum, thresholds, action and quantization remain required for scale evolution.

6 A positive candidate contract and its exit gates

The candidate branch has complete finite *matter* kinematics: choose a fixed tetrahedron, rank r, unitary links $U(s)$, local temporal blocks $\Omega(s)$, initial $c(0)$ and readouts (5),(7). It has well-defined norm-preserving linear evolution and full family relabeling covariance. On the curved branch D_U is a covariant transport operator, not the original topological boundary.

The independent link variables currently have no source-selected evolution. A nonnegative gauge-invariant potential can, for example, be *proposed* as $V = \sum_d \kappa_d [1 - \mathrm{Re}\,\mathrm{tr}\, W_d/r]$, $\kappa_d > 0$. This is a supplied Wilson-type choice, not a derived gauge action. A potential alone does not define a link clock, canonical momenta, Gauss constraint, feedback or quantum measure. Symmetry may constrain some coefficient classes without fixing their values.

Admissibility requirement	Outcome for this branch
Explicit states, fibers and transport	Defined, conditional on supplied r, U, Ω.
Local gauge and signed S_4 family covariance	Proved; configuration invariance is distinct.
Conserved support and regional current law	Proved, including curved transport.
Original chain identity and Hodge square	Preserved only on the flat branch; proved scope.
Independent curvature readout	Defined; acquisition and physical interpretation open.
Closed gauge-field dynamics and Gauss law	Open; needs a joint action or phase-space law.
Continuum/Lorentz and chiral matter limit	Open; no convergence transferred from algebra.
SM group, charges, anomalies, normalization	Open; no native selection is supplied.
RG running, proton channels, monopole sector	Downstream of the missing field/action bridge.

Discrete connections and the curvature/nilpotency relationship are established [4]. Algebraic covariance alone is not a convergence theorem: other discretizations specifically investigate convergence [5]. Our Hasse-link construction is not identified with their smooth discretizations or their continuum limit.

6.1 Conditional analogue experiment specification

A bounded operator analogue of Corollary 2 would require: 15 calibrated complex modes, the 28 prescribed signed incidence couplings, matched coupling magnitudes, a controlled phase on the chosen incidence and a known common rate $\eta > 0$. In a proposed Schrodinger-type realization use $H_{\text{phys}} = i\hbar\eta A_U$; then the evolution is $\dot{c} = \eta A_U c$ in laboratory time. At phases 0 and $\pi/2$, the predicted nonzero frequency sets are respectively $\{2\eta\}$ and $\{\sqrt{2}\eta, 2\eta, \sqrt{6}\eta\}$ with the multiplicities in (9). Measured complex transfer matrices or several calibrated preparations are preferable to a single power-spectrum trace that may miss modes.

Before testing, freeze the mapping, phase uncertainty, hopping errors, readout noise/covariance, frequency estimator, time window, null model, acceptance threshold, exclusions and retained data. Gauge-related rephasings are the equivalence control; uniform magnitude changes are a distinct confound. This is a model-to-engineered-operator candidate, not a test of GUT unification or proton/monopole physics. No physical experiment or dataset fit has been performed.

7 Verification, provenance and review

The fresh primary replay `verify_hasse_transport.py` constructs all simplices and incidence signs, enumerates 18 diamonds, checks the 14-dimensional scalar cycle constraint, tests non-Abelian rank-two covariance, all 24 signed relabelings, temporal covariance, local support balance and the phase-dependent characteristic polynomial on five exact phases. The spectral proof above uses a separate two-plane Gram reduction, not the computed polynomial as a universal proof. The companion `verify_gauge_bridge.py` independently checks native centralizer dimensions, state-phase flatness, conditional $SU(5)$ trace normalization and an ordered-vertex transport comparison. That comparison is an additional convention, not the Hasse-link model and not a claim of its full permutation covariance.

The receipts record exact rational, Gaussian-rational and algebraic arithmetic. They are constructor checks, not independent review. No source script was executed merely to ingest an

archive. Source baseline: Public Atlas commit `84d6636cbf7d5dcb3c8c9e8abbe7b43350908499`; UD repository baseline `2a6d3fda46dd4724aeba73329b605e6533d68d5d`. The historical v0.39 ledger is preserved alongside later source amendments. The new objects and conditional conclusions do not change frozen physical status or claim confidence by citation. The unavailable Genesis image is not used.

Author responsibility and independent mathematical review remain submission tasks. AI assistance includes proof drafting, exact fixtures, source comparison and document preparation. No claim of new general gauge theory, funding status, experimental discovery or physical promotion is made.

References

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